Simple, Regular, and Efficient Numerical Integration of Rotation

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Summary

- All-Purpose Method to Integrate Rotation
  - 7 Variables $(q_0, q_1, q_2, q_3; \omega_A, \omega_B, \omega_C)$
  - Constraint: Renormalization

$$q_j \to \frac{q_j}{\sqrt{\sum_{k=0}^{3} q_k^2}}$$

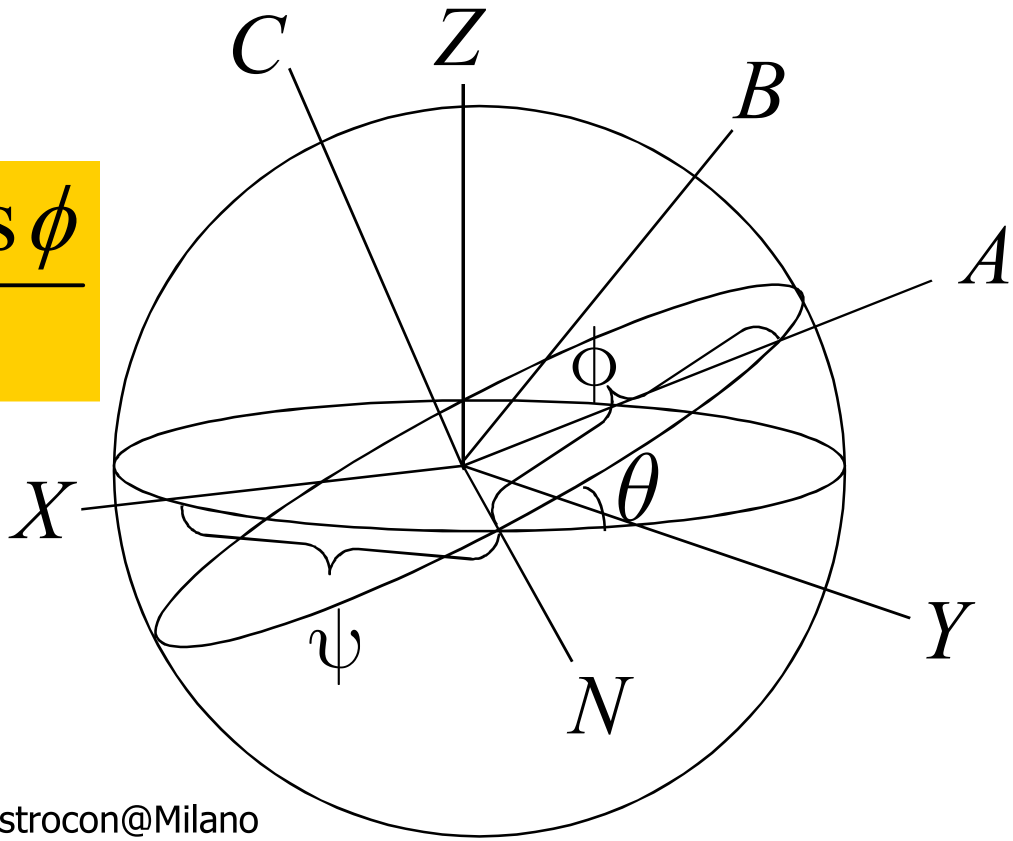
- Characteristics
  - Simple: Eq. of Motion is Quadratic
  - Regular: No Singularity
  - Efficient: Fastest & Least Error

Defect of Euler Angles

- 3-1-3 Euler Angles (ψ, θ, ϕ)
- Small Obliquity

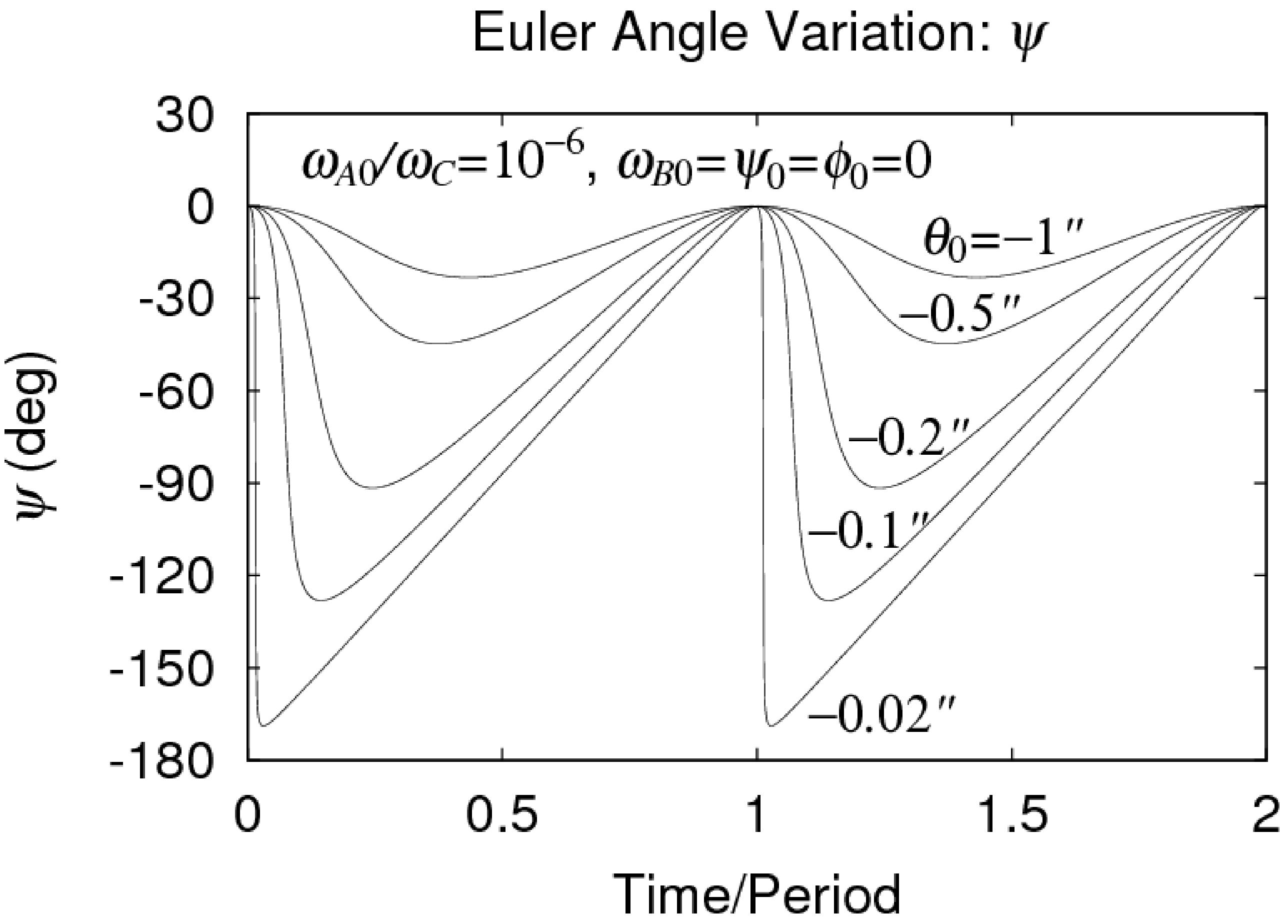
$$\frac{d\psi}{dt} = \frac{\omega_A \sin\phi + \omega_B \cos\phi}{\sin\theta}$$

- Same for Other Conventions



Example of Defects

- Equatorial CS
- Rigid Earth
- Torque-Free
- Small PM
- Peculiar Expression
- Close Encounter of C- and Z-axis



Euler Parameters

- Unit Quarternion (q_0, q_1, q_2, q_3)
  - No Trigonometric Functions

$$\begin{pmatrix} (\mathbf{e}_A)^{\mathrm{T}} \\ (\mathbf{e}_B)^{\mathrm{T}} \\ (\mathbf{e}_C)^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_0 q_3 + q_1 q_2) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_0 q_1 + q_2 q_3) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

- Redundant Variables
  - Normalization $|q|^2 \equiv q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

Body-Fixed Angular Velocity Components

- Body-Fixed CS $\left(\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_C\right)$
- Angular Velocity Components $\left(\omega_A, \omega_B, \omega_C\right)$
- Dynamical Equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix} = \begin{pmatrix}-\alpha\omega_B\omega_C + aN_A \\ -\beta\omega_C\omega_A + bN_B \\ -\gamma\omega_A\omega_B + cN_C\end{pmatrix}$$

$$\alpha = \left(C - B\right)/A$$

$$\beta = \left(A - C\right)/B$$

$$\gamma = \left(B - A\right)/C$$

$$a = 1/A$$

$$b = 1/B$$

$$c = 1/C$$

Kinematical Equation

- Quite Simple

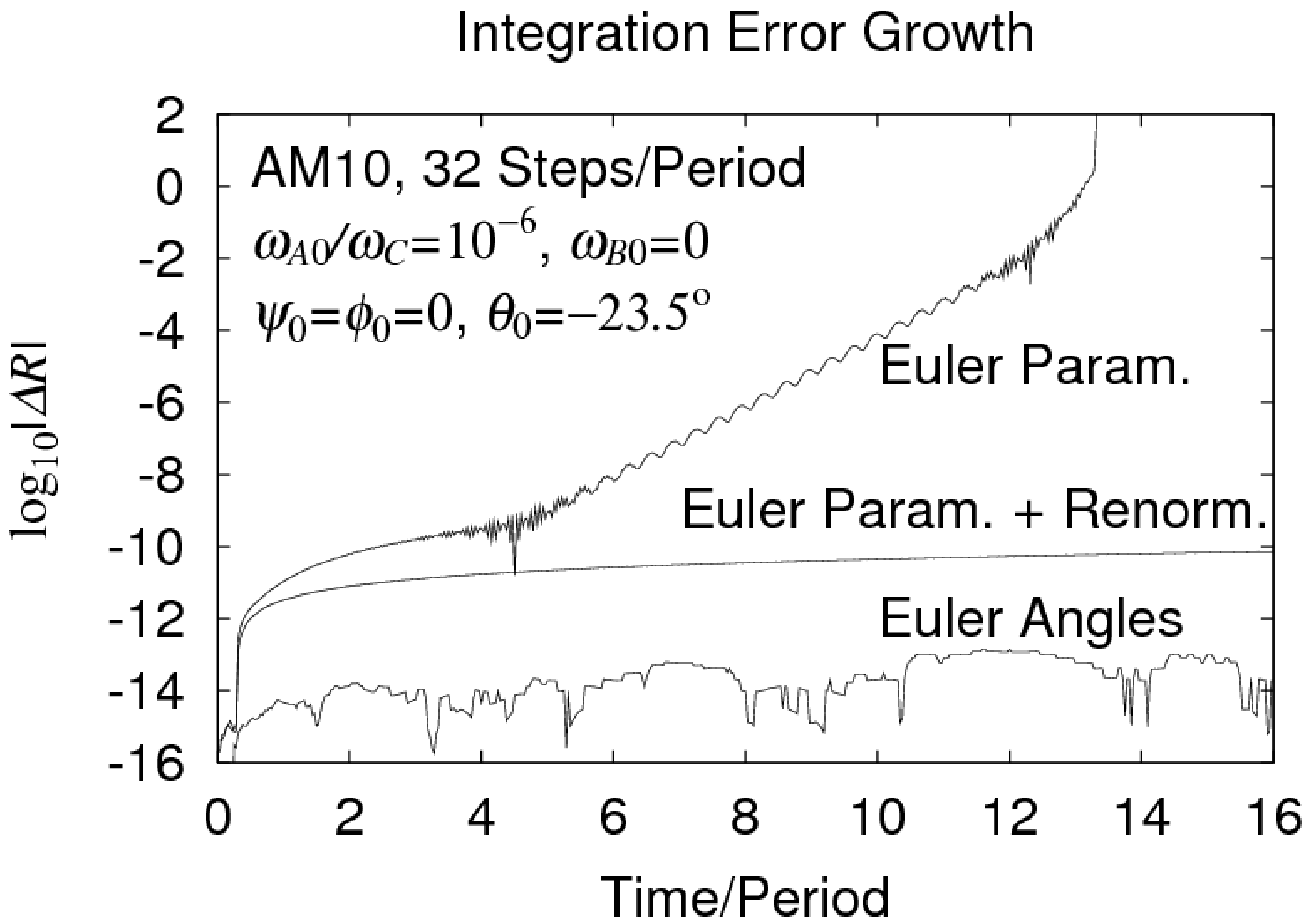
$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_A & -\omega_B & -\omega_C \\ \omega_A & 0 & \omega_C & -\omega_B \\ \omega_B & -\omega_C & 0 & \omega_A \\ \omega_C & \omega_B & -\omega_A & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Redundant Variables

- Indefinite Inverse Transformation
  - Complicated Partial Derivatives
  - Difficulty in Data Fitting
- Numerical Instability
  - Ex.: Altman Variables (Altman, 1976)
  - Euler Parameters + Angular Momentum Vector

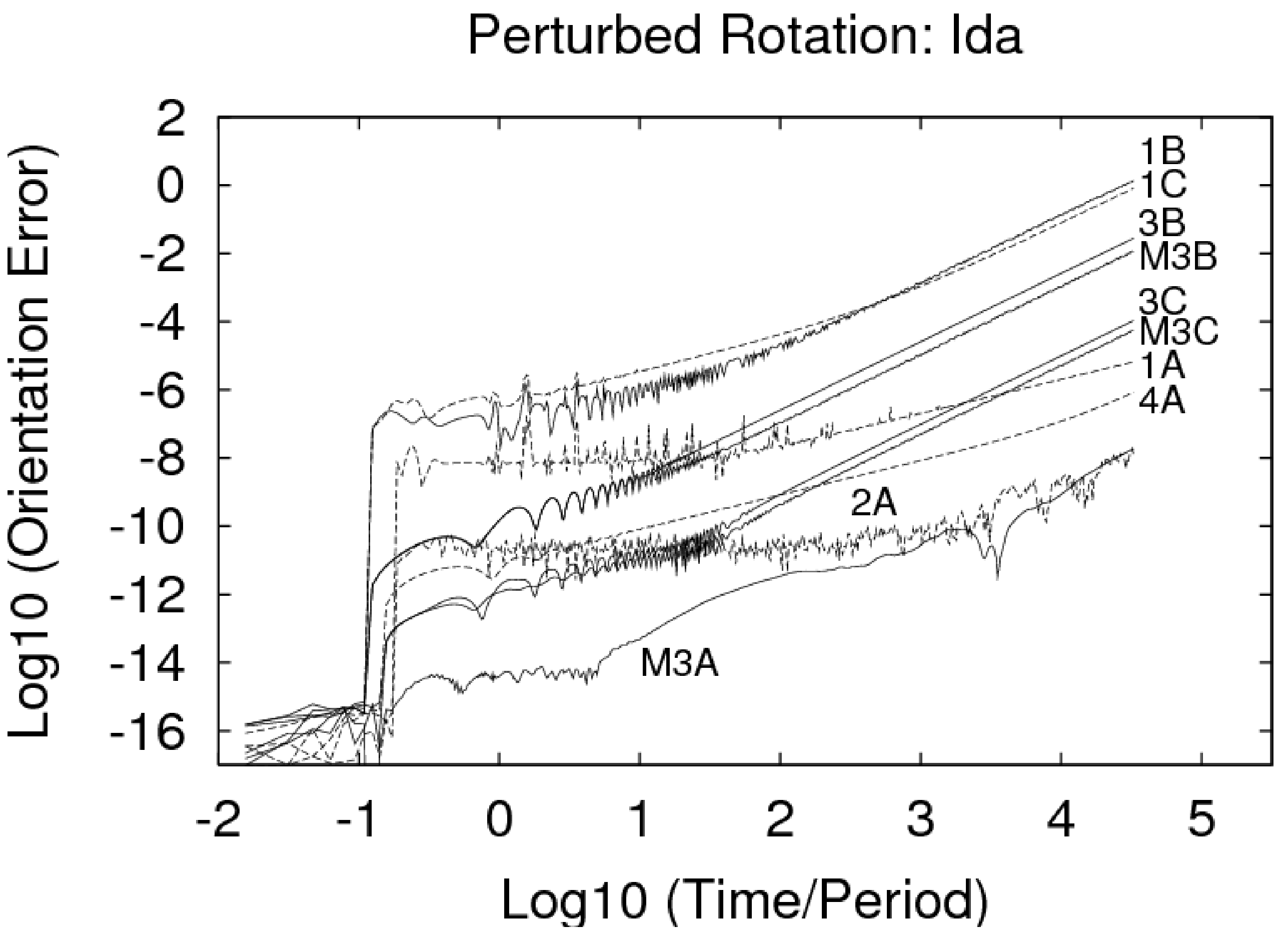
$$\left(q_0, q_1, q_2, q_3; L_X, L_Y, L_Z\right)$$

Instability of Altman V.



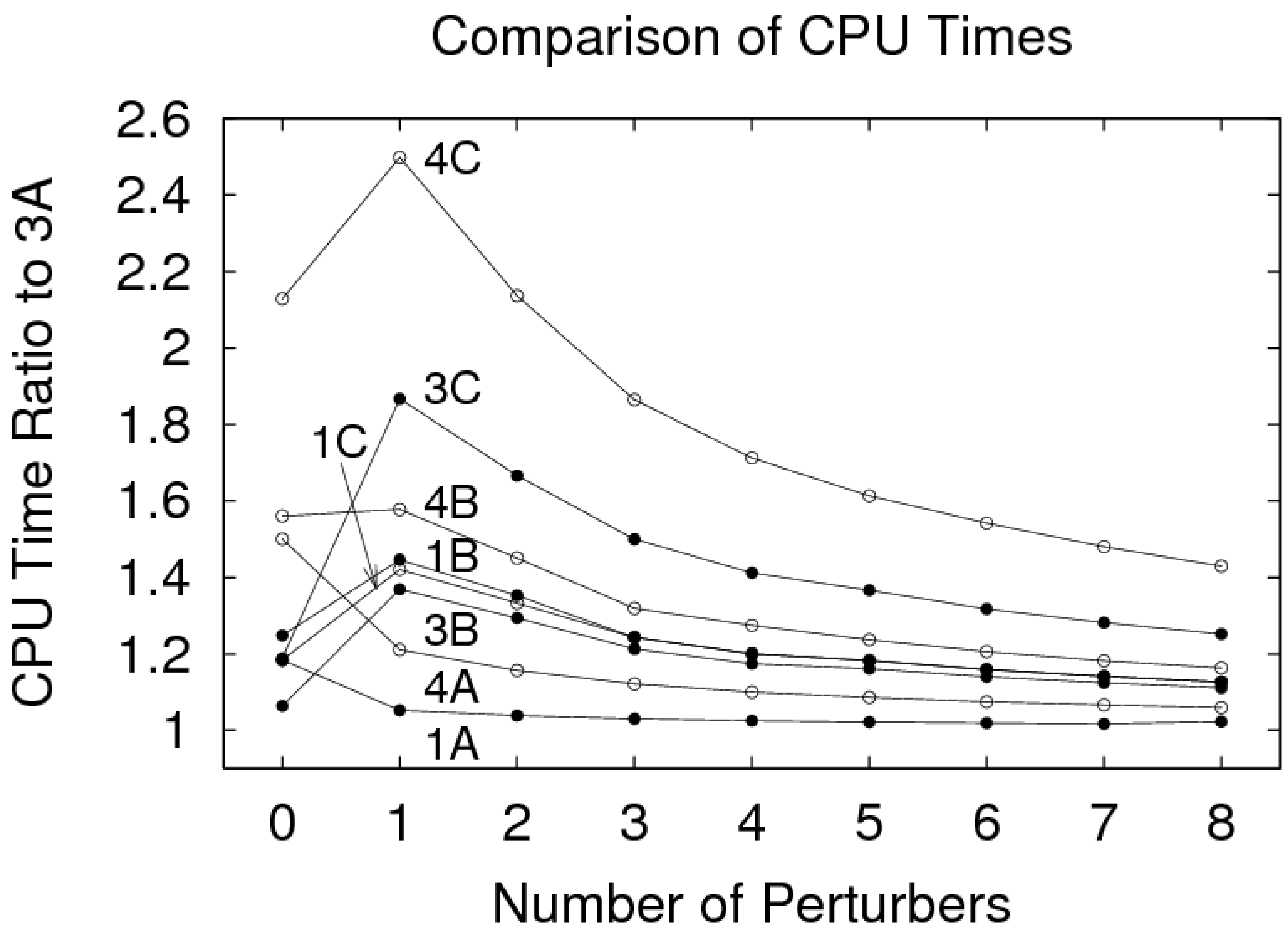
Integration Error

- Ecliptic CS
- Asteroid Ida
- Solar Torque
- M3A: New
- 2A: 1-2-3 EA
- 4A: DCM
- 1A: 3-1-3 EA
- 3B: Altman



CPU Time

- 3A: New
- 1A: 3-1-3 EA
- 4A: DCM
- 3B: Altman



Conclusion

- All-Purpose Method
  - Simple: Plain Implementation
  - Regular: Arbitrary Triaxiality, Initial Cond.
  - Efficient: Fastest & Highest Precision
- Encke's Method: Quasi-Uniform Rotation
- Open Question
  - Treatise in Analytical Dynamics