

# Validated Integration of Solar System Dynamics

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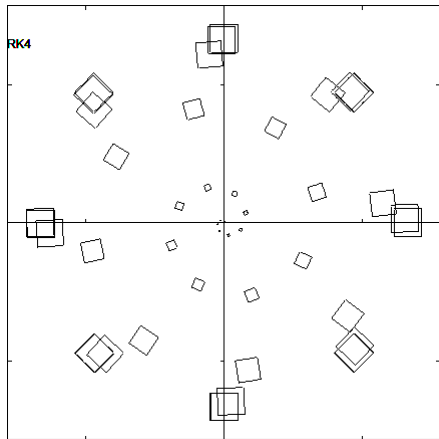
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# Motivation for Verified Numerics

- Numbers Smaller than Machine Epsilon
  - $\epsilon < \epsilon_m$
  - $((((1 + \epsilon) + \epsilon) + \epsilon) + \dots$

## Motivation for Verified Numerics

- Numbers Smaller than Machine Epsilon
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  - $((1 + \epsilon) + \epsilon) + \dots$
- Example of RK Errors - Harmonic Oscillator with too few steps
  - $x_1' = x_2$
  - $x_2' = -x_1$



# Intervals

Popular Codes: AWA VNODE

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a, b] + [-d, -c] = [a - d, b - c]$$

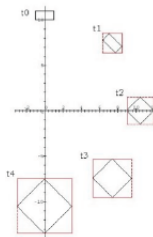
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- Overestimations:
- Wrapping Effect  
Primarily caused by  
enclosures of non-convex  
sets attributable to  
nonlinearities



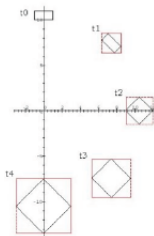
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- Dependency Problem

$$x = [a, b]$$

$$x - x = [a, b] - [a, b] = [a, b] + [-b, -a] = [a - b, b - a]$$

# Differential Algebras

- Differential Algebra

$$\text{First Order } {}_1D_1 (a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1)$$

$$t \cdot (a_0, a_1) = (t \cdot a_0, t \cdot a_1)$$

$$(a_0, a_1) \cdot (b_0, b_1) = (a_0 \cdot b_0, a_0 \cdot b_1 + a_1 \cdot b_0)$$

$$(a_0, a_1)^{-1} = (1/a_0, -a_1/a_0^2)$$

- Automatic Differentiation - Griewank 1989
- Extendable to  ${}_nD_v$

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- Truncated Power Series Algebra

Compute Value and Derivative at 1 of  $f(x) = x^2$

$$[x] = (1, 1) = 1 + \delta x$$

$$f([x]) = [x]^2 = (1 + \delta x)^2 = 1 + 2\delta x + \delta x^2$$

# ODE Flows

- ODE initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

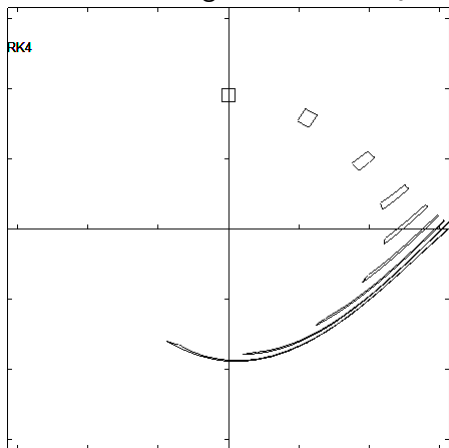
- Explicit Euler

$$x_{k+1} = x_k + f(x_k) \cdot h$$

- Naive replacement:

$$x_0 \rightarrow [x_0] = (x_0, 1)$$
$$[x]_{k+1} = [x]_k + f([x]_k) \cdot h$$

Pendulum - Runge Kutta 4 -  $2D_9$



# Taylor Models

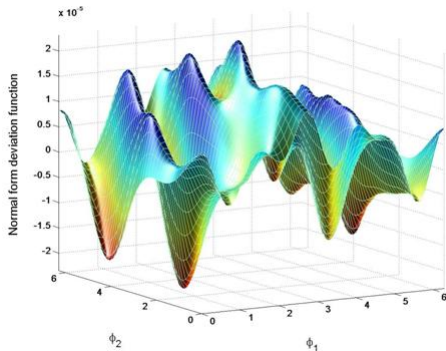
- Differential Algebra for functional dependence+  
Intervals for rigorous bounding (Remainder Bound)
  - Taylors Theorem
  - $f(x) = P_{n,x_0,f} + \frac{1}{(n+1)!} ((x - x_0) \cdot \nabla)^{n+1} f(x_0 + (x - x_0)\theta)$   
For:  $\theta \in \mathbb{R}, 0 < \theta < 1$
  - Allows us to obtain rigorous error estimates
- Truncation terms in the Power Series Algebra Swept into  
Remainder Bound

# Taylor Model Integration

- Rewrite differential equation as integral equation:  
$$\vec{r}(t) = \vec{r}_0 + \int_t^{t_0} \vec{F}(\vec{r}(t'), t') dt'$$
- Introduce A:  $A(\vec{f})(t) = \vec{r}$  Transforming problem to a fixed point problem
- Employ Schauder's Fixed Point Theorem.

# COSY Infinity

Developed originally for computing the Normal Form Defect Function



Lots of Dependency Problems

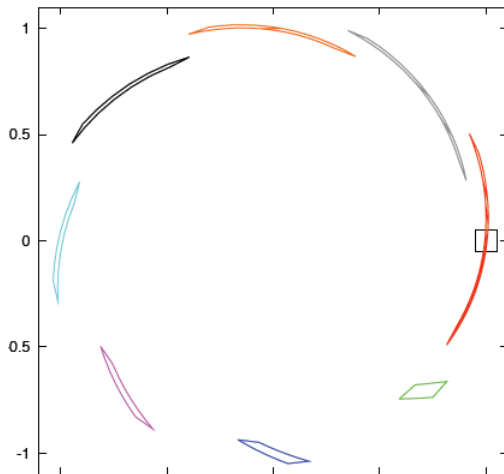
# COSY Verified Integrator

## COSY-VI

- Shrink Wrapping  
Method for reducing remainder bound by increasing Polynomial Part.  
ODE Dependent
- Preconditioning  
Appropriate coordinate transformation, also applied to Interval Systems
- Blunting  
Procedure for Ill-conditioned Matrices  $A(\vec{f})$
- To be released: Dynamic domain decomposition  
Splitting of flow representation to avoid high nonlinearities

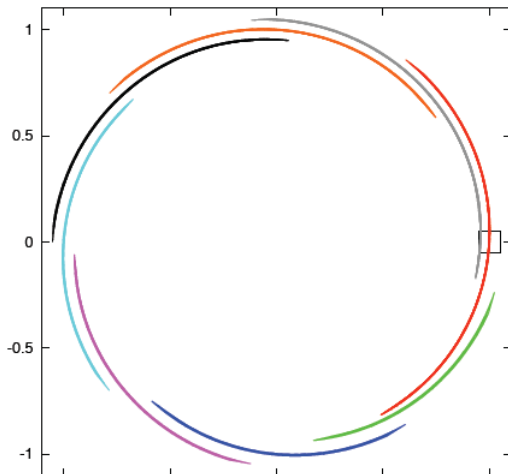
# Taylor Models at work in the Discrete Kepler

discrete kepler. 1st revolution, ICw 0.1, NO=13 w7



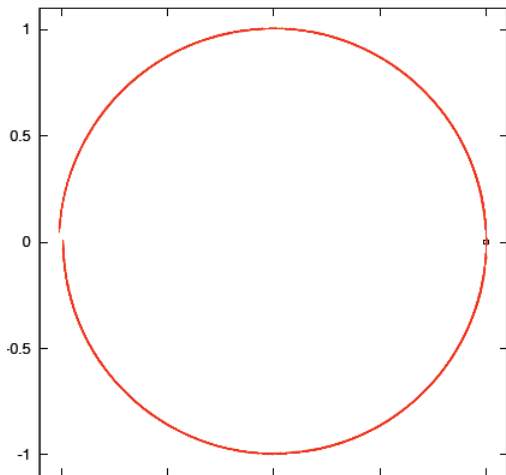
## Taylor Models at work in the Discrete Kepler cont.

discrete kepler. 2nd revolution, ICw 0.1, NO=13 w7



## Taylor Models at work in the Discrete Kepler cont. 2

discrete kepler. 33rd revolution, ICw 0.02, NO=13 w7



# The Problem

- The ODE - Restricted n-body problem

$$\frac{d^2\vec{r}}{dt^2} = G \sum_i m_i \frac{\vec{r}_i - \vec{r}}{|\vec{r}_i - \vec{r}|^3}$$

- Interval Methods, study done in 2004 - Politecnico di Milano
  - Not Suitable for long term Studies of motion
  - Taylor Model Methods worth looking into

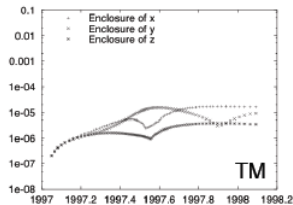
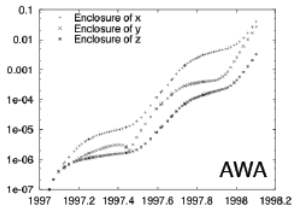
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- JPL DE405 + Cubic Spline Interpolation
- Relativistic Effects
- Initial conditions corresponding to current measurement uncertainties

# Test Cases

- XF11

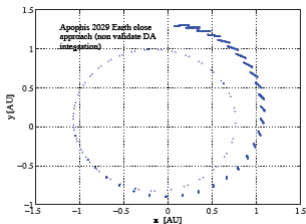
Proof of concept: Done by J. Hoefkens (2001)



<i>Date</i>	<i>Min. Dist. (AU)</i>
October 31, 2002	0.064
June 10, 2016	0.180
October 26, 2028	0.006
June 8, 2035	0.174
November 4, 2040	0.150

# Apophis

- Shoemaker Levi 9
- Apophis
  - Integration Via non-verified Means agrees with JPL Horizons system to  $1e-7$  AU 15km over 20 years.







## Future Works

- ODE Modifications:
- Chebychev Polynomial Interpolation of DE405
- Absorbtion and Reflection for anisotropic Distributions
- Inclusion of Yarkovsky Effect
- Inclusion of Interval Bounds for other small effects
- Near Earth Encounter - Apophis 2029 / 2036  
Part of Ariadne Project
- Computation of Impact Leading Conditions  
Via Rigorous Map Inversion
- Analysis of proposed Asteroid Deflection Techniques

Fin

Thank you for your time!  
Any Questions?

-  The Nautical Almanac Office, U.S. Naval Observatory with contributions from H.M. Nautical Almanac Office, Royal Greenwich Observatory, Jet Propulsion Laboratory, Bureau Des Longitudes, and The Time Service and Astrometry Departments, U.S. Naval Observatory *Explanatory Supplement to the Astronomical Almanac*. University Science Books, 1992.
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*stabilization by shrink wrapping* International Journal of Differential Equations and Applications, 2006.



K. Makino *Suppression of the wrapping effect by Taylor model-based verified integrators: Long-term stabilization by preconditioning*, International Journal of Differential Equations and Applications, 2006.